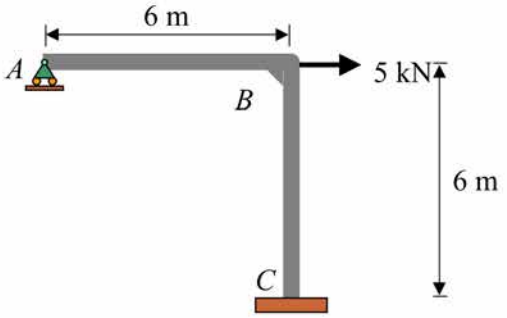
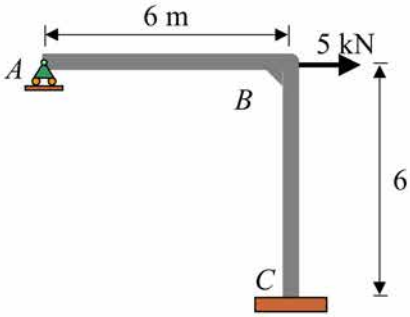
Example 1

For the frame shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.
(c) Draw the **quantitative shear** and **bending moment diagrams.**
$E = 200$ GPa, $I = 60(10^6)$ mm^4, $A = 600$ mm^2

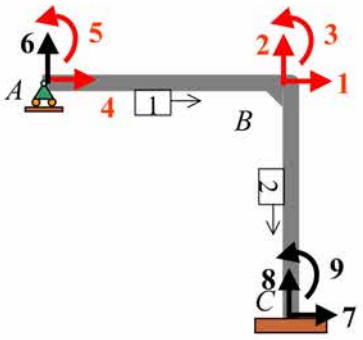




$$\frac{AE}{L} = \frac{(600 \times 10^{-6}\,\mathrm{m}^2)(200 \times 10^{6}\,\frac{\mathrm{kN}}{\mathrm{m}^2})}{6\mathrm{m}} = 20000\ \mathrm{kN/m}$$

$$\frac{12EI}{L^3} = \frac{12(200 \times 10^{6}\,\frac{\mathrm{kN}}{\mathrm{m}^2})(60 \times 10^{-6}\,\mathrm{m}^4)}{(6\mathrm{m})^3} = 666.667\ \mathrm{kN/m}$$

Global :



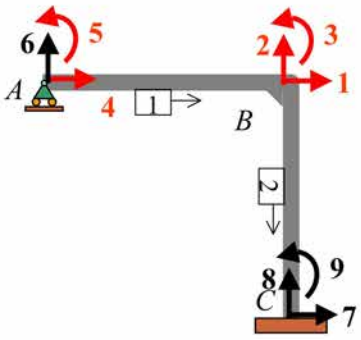
$$\frac{6EI}{L^2} = \frac{6(200 \times 10^{6}\,\frac{\mathrm{kN}}{\mathrm{m}^2})(60 \times 10^{-6}\,\mathrm{m}^4)}{(6\mathrm{m})^2} = 2000\ \mathrm{kN}$$

$$\frac{4EI}{L} = \frac{4(200 \times 10^{6}\,\frac{\mathrm{kN}}{\mathrm{m}^2})(60 \times 10^{-6}\,\mathrm{m}^4)}{6\mathrm{m}} = 8000\ \mathrm{kN \bullet m}$$

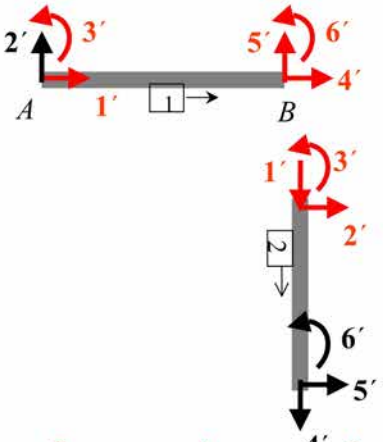
$$\frac{2EI}{L} = \frac{2(200 \times 10^{6}\,\frac{\mathrm{kN}}{\mathrm{m}^2})(60 \times 10^{-6}\,\mathrm{m}^4)}{6\mathrm{m}} = 4000\ \mathrm{kN \bullet m}$$

Using Transformation Matrix:

Global :



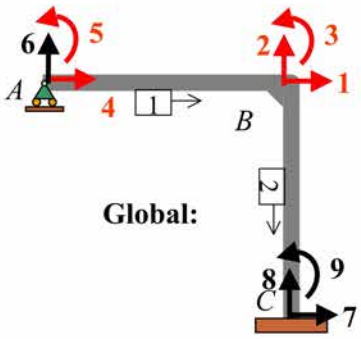
Local :



• **Member Stiffness Matrix**

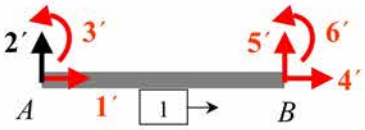
$$[k'] = \begin{array}{c} \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

Stiffness Matrix: Member 1



Global:

Local :

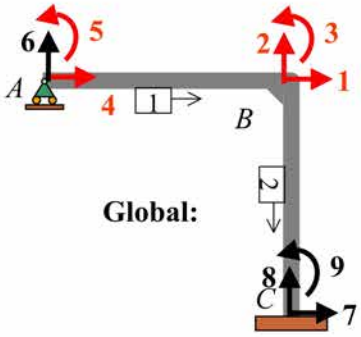


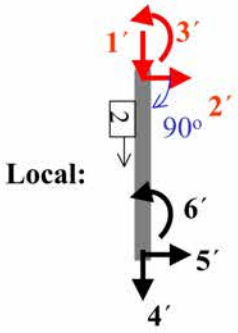
$[q] = [q']$

$\rightarrow [k]_1 = [k']_1$

$$[k]_1 = \begin{array}{c} \\ 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix} \end{array}$$

Stiffness Matrix: Member 2





$\lambda_{ix} = \cos(-90^o) = 0$
$\lambda_{iy} = \sin(-90^o) = -1$

$\lambda_{jx} = \cos(-90^o) = 0$
$\lambda_{jy} = \sin(-90^o) = -1$

$$[q]_2 = [T]^T [q']_2$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{pmatrix}$$

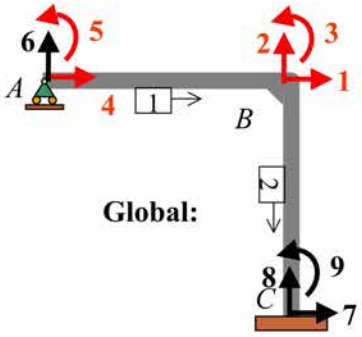
$[T]^T$

$$[k']_2 = \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix} \end{array}$$

$$[k]_2 = [T]^T [k']_2 [T]$$

$$[k]_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 666.667 & 0 & 2000 & -666.667 & 0 & 2000 \\ 0 & 20000 & 0 & 0 & -20000 & 0 \\ 2000 & 0 & 8000 & -2000 & 0 & 4000 \\ -666.667 & 0 & -2000 & 666.667 & 0 & -2000 \\ 0 & -20000 & 0 & 0 & 20000 & 0 \\ 2000 & 0 & 4000 & -2000 & 0 & 8000 \end{pmatrix} \end{array}$$

Global Stiffness Matrix:



$[k]_1$

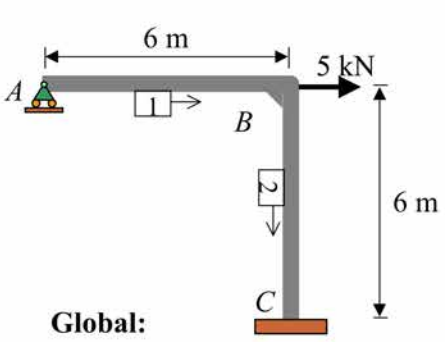
	4	6	5	1	2	3
4	20000	0	0	-20000	0	0
6	0	666.667	2000	0	-666.667	2000
5	0	2000	8000	0	-2000	4000
1	-20000	0	0	20000	0	0
2	0	-666.667	-2000	0	666.667	-2000
3	0	2000	4000	0	-2000	8000

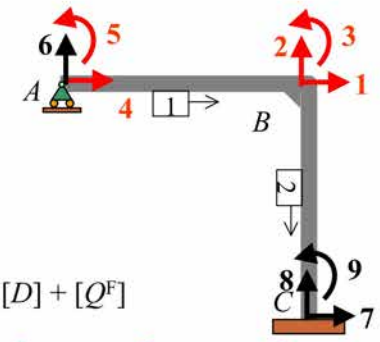
$[k]_2$

	1	2	3	7	8	9
1	666.667	0	2000	666.667	0	2000
2	0	20000	0	0	-20000	0
3	2000	0	8000	-2000	0	4000
7	-666.667	0	-2000	666.667	0	-2000
8	0	-20000	0	0	20000	0
9	2000	0	4000	2000	0	8000

$[K]$

	4	5	1	2	3
4	20000	0	-20000	0	0
5	0	8000	0	-2000	4000
1	-20000	0	20666.667	0	2000
2	0	-2000	0	20666.667	-2000
3	0	4000	2000	-2000	16000



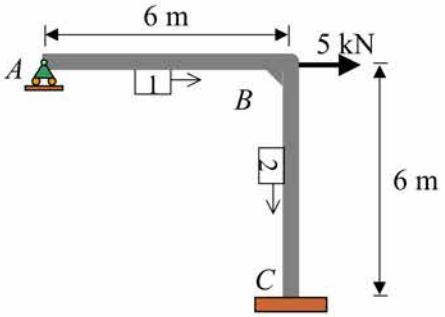


$[Q] = [K][D] + [Q^F]$

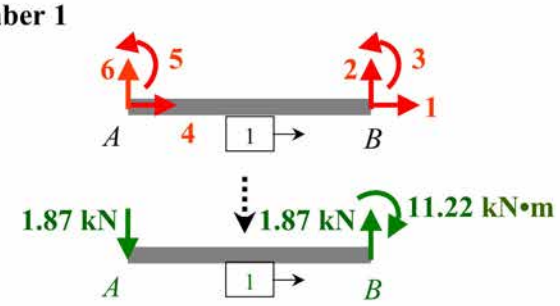
Global:

$$\begin{pmatrix} Q_4 = 0 \\ Q_5 = 0 \\ Q_1 = 5 \\ Q_2 = 0 \\ Q_3 = 0 \end{pmatrix} = \begin{matrix} 4 \\ 5 \\ 1 \\ 2 \\ 3 \end{matrix} \begin{pmatrix} 20000 & 0 & -20000 & 0 & 0 \\ 0 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 20666.667 & 0 & 2000 \\ 0 & -2000 & 0 & 20666.667 & -2000 \\ 0 & 4000 & 2000 & -2000 & 16000 \end{pmatrix} \begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0.01316 \text{ m} \\ 9.199(10^{-4}) \text{ rad} \\ 0.01316 \text{ m} \\ -9.355(10^{-5}) \text{ m} \\ -1.887(10^{-3}) \text{ rad} \end{pmatrix}$$

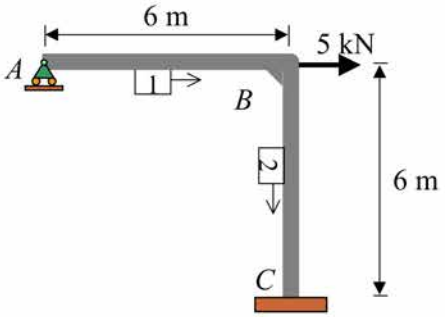


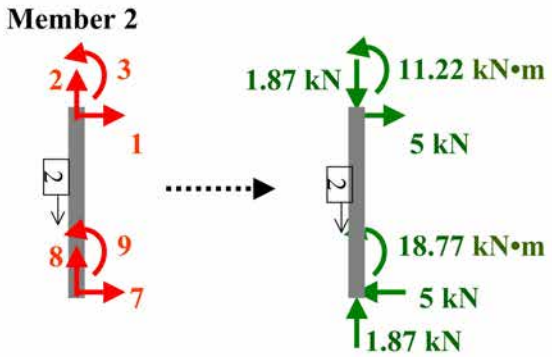
Member 1



$$[q]_1 = [k]_1[d]_1 + [q^F]_1$$

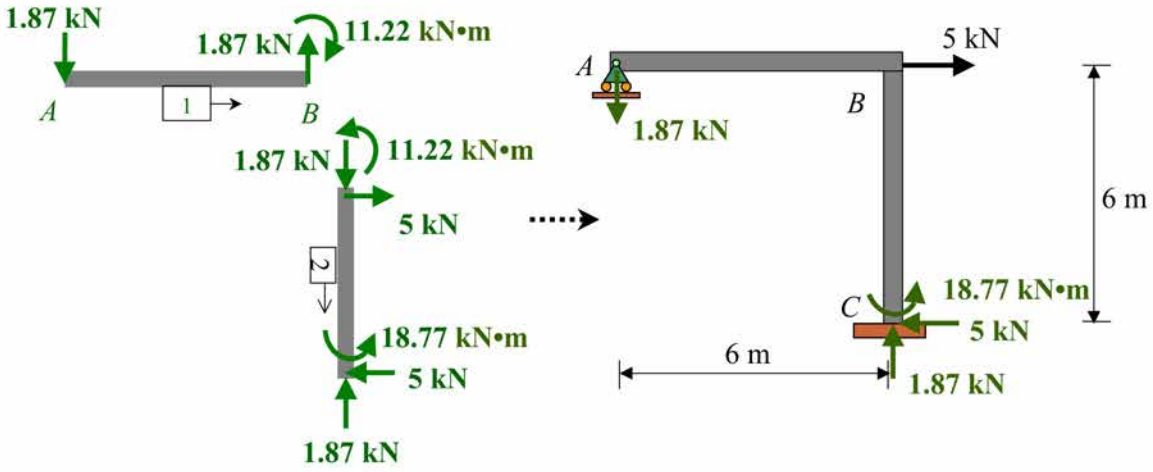
$$\begin{pmatrix} q_4 \\ q_6 \\ q_5 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{array}{c} \\ 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 20000 & 0 & 0 & -20000 & 0 & 0 \\ 0 & 666.667 & 2000 & 0 & -666.667 & 2000 \\ 0 & 2000 & 8000 & 0 & -2000 & 4000 \\ -20000 & 0 & 0 & 20000 & 0 & 0 \\ 0 & -666.667 & -2000 & 0 & 666.667 & -2000 \\ 0 & 2000 & 4000 & 0 & -2000 & 8000 \end{pmatrix} \end{array} \begin{pmatrix} D_4 = 0.01316 \\ D_6 = 0 \\ D_5 = 9.199(10^{-4}) \\ D_1 = 0.01316 \\ D_2 = -9.355(10^{-5}) \\ D_3 = -1.887(10^{-3}) \end{pmatrix} = \begin{pmatrix} 0 \\ -1.87 \\ 0 \\ 0 \\ 1.87 \\ -11.22 \end{pmatrix}$$

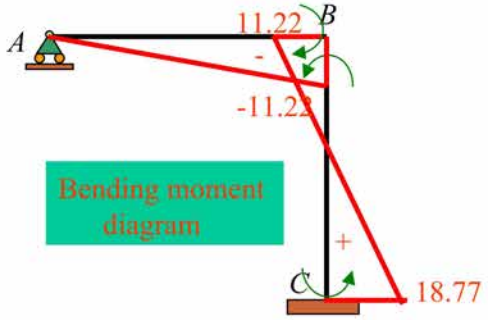


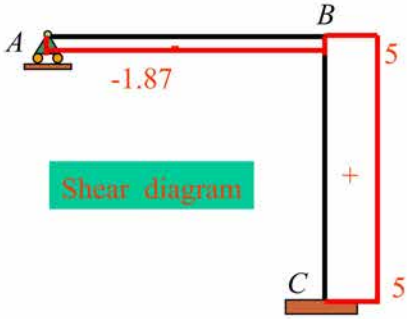


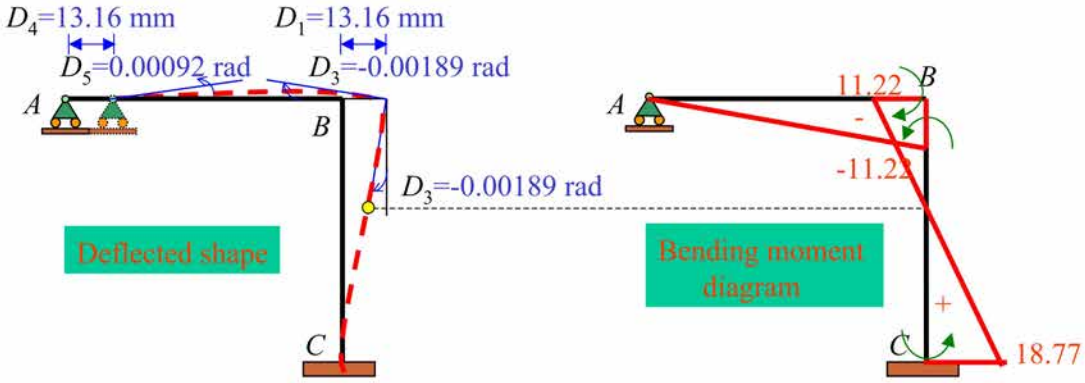
$$[q]_2 = [k]_2[d]_2 + [q^F]_2$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 666.667 & 0 & 2000 & -666.667 & 0 & 2000 \\ 0 & 20000 & 0 & 0 & -20000 & 0 \\ 2000 & 0 & 8000 & -2000 & 0 & 4000 \\ -666.667 & 0 & -2000 & 666.667 & 0 & -2000 \\ 0 & -20000 & 0 & 0 & 20000 & 0 \\ 2000 & 0 & 4000 & -2000 & 0 & 8000 \end{pmatrix} \end{array} \begin{pmatrix} D_1 = 0.01316 \\ D_2 = -9.355(10^{-5}) \\ D_3 = -1.887(10^{-3}) \\ D_7 = 0 \\ D_8 = 0 \\ D_9 = 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -1.87 \\ 11.22 \\ -5 \\ 1.87 \\ 18.77 \end{pmatrix}$$


1.87 kN
1.87 kN
11.22 kN•m
A
1
B
1.87 kN
11.22 kN•m
5 kN
2
18.77 kN•m
5 kN
1.87 kN
A
5 kN
B
1.87 kN
6 m
C
18.77 kN•m
5 kN
6 m
1.87 kN


A
11.22
B
-
-11.22
+
C
18.77
Bending moment diagram


A
B
-1.87
5
+
C
5
Shear diagram



$$\begin{pmatrix} D_4 \\ D_5 \\ D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0.01316 \text{ m} \\ 9.199(10^{-4}) \text{ rad} \\ 0.01316 \text{ m} \\ -9.355(10^{-5}) \text{ m} \\ -1.887(10^{-3}) \text{ rad} \end{pmatrix}$$